On Anomalies in the Intensity in Diffracted Spectra.

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Professor Wood[*] has found that the spectra of an incandescent lamp formed by a grating can have bright and dark lines in them. These occur at the same place in each order of spectrum, and move through the spectrum when the inclination of the grating is altered. The phenomenon has been investigated by Rayleigh,[†] who discusses the case of diffraction by a grooved reflecting surface. On account of the analytical difficulties he uses approximations. In the present paper a theoretical grating is discussed, the phenomena of which can be investigated accurately, an advantage that is, however, coupled with the drawback that we cannot feel certain how far the phenomena agree with those of the ordinary metallic reflection grating.

A theorem that gives the sum of certain infinite series in a finite form, and transforms others into new series, is found in § 5.

2. Let the grating consist of an infinite number of very thin rods of specific inductive capacity unity, each electrified to a line density σ and capable of oscillating, lying at equal distances a parallel to each other in a plane. We take the axis of z along one of the rods, that of x in the plane of the grating, and that of y perpendicular to that plane.

Let the incident waves be propagated in a direction parallel to the plane of xy and making an angle θ with the axis of y, and let the electric force be

$$\exp ip\,(t - y\cos\theta - x\sin\theta).$$

The displacement of a rod is

$$\delta = \mathrm{A}\exp ip\,(t - x\sin\theta),$$

where A is complex, unless the phase of the displacement of the rod is the same as that of the wave incident on it.

Since the diffracted waves depend only on the motion of the rods, and are given by a formula without peculiarities, the features that characterise these waves will depend only on the behaviour of the quantity A.

3. We first consider the case where the electric vibration in the incident waves is parallel to the rods. The rods here vibrate longitudinally. The forces acting on one of them are: i, a force of restitution partly of mechanical origin and partly due to the repulsions of the other rods, say, $b\delta$; ii, a force of

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[*] R. W. Wood, 'Phil. Mag.,' Ser. 6, vol. 4, 1902, p. 396.

[†] Lord Rayleigh, 'Roy. Soc. Proc.,' A, vol. 79, 1907, p. 399.

electrical origin due to the rod itself and in phase with the motion, which produces an apparent increase in the inertia of the rod; iii, a similar force in quadrature with the motion; iv, another force, due to the action of all the other rods. The work done by the third force is equal to the energy radiated by an isolated rod, and hence the force is easily shown to be of the form $-ikAp^2\sigma^2$, where k is a constant.

4. In order to find the value of the last force, we must find the field due to the oscillating rods. Now, the components of the electric force at (x, y, z) due to an oscillating charge $\sigma d\zeta$ situated at $(0, 0, \zeta)$ are

$$X = d^2\Pi/dx\,dz, \qquad Y = d^2\Pi/dy\,dz, \qquad Z = d^2\Pi/dz^2 + p^2\Pi,$$

where

$$\Pi = A\sigma d\zeta e^{-ip\rho}/\rho,$$

ρ being the distance of (x, y, z) from $(0, 0, \zeta)$. On integrating with respect to ζ, we find that X, Y, and the first term of Z vanish, and we have

$$Z = 2A\sigma p^2 \int_0^\infty \Pi d\zeta = 2A\sigma p^2 \int_0^\infty d\rho e^{-ip\rho}/\sqrt{(\rho^2 - r^2)},$$

or if $\rho = ru$,

$$= 2A\sigma p^2 \int_1^\infty du\, e^{-ipru}/\sqrt{(u^2-1)} = A\sigma p^2\pi\{-iJ_0 pr - \Upsilon_0 pr\},*$$

where Υ_0 is that second solution of Bessel's equation, the graph of which at infinity consists of waves of the same amplitude as those of the Bessel's function, but in quadrature with them. Hence the mechanical force exerted on the rod by the other rods is

$$\Sigma A\sigma^2 p^2 \pi\{-iJ_0 pr - \Upsilon_0 pr\} 2\cos(px \sin\theta)$$
$$= -2A\sigma^2 p^2 \pi \sum_1^\infty \{iJ_0\, npa + \Upsilon_0\, npa\} \cos(npa \sin\theta),$$

since $r = x = na$.

5. Consider the expression

$$\tfrac{1}{2}\int_0^\infty f(\lambda)\{1 + \sum_1^\infty c^n \cos(\lambda - \alpha) nu + \sum_1^\infty c^n \cos(\lambda + \alpha) nu\} d\lambda.$$

Its limit when $c = 1$ is

$$\tfrac{1}{2}\int_0^\infty f(\lambda)\, d\lambda + \sum_1^\infty \cos n\alpha u \int_0^\infty f(\lambda) \cos n\lambda u\, d\lambda. \qquad \text{(A)}$$

Also in the limit, $\frac{1}{2} + \Sigma c^n \cos(\lambda - \alpha) nu$ vanishes unless $(\lambda - \alpha) u$ is a multiple of 2π (say $2p\pi$) and its integral with respect to λ from a value slightly less to one slightly greater than one of the critical values is π/u. Hence the series also is

$$(\pi/2u) \Sigma [f(2m\pi/u + \alpha) + f(2m\pi/u - \alpha)], \qquad \text{(B)}$$

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* Gray and Mathews, 'Bessel's Functions,' p. 230, Ex. 18.

where only positive values of the argument are to be taken. The function $f(\lambda)$ need not be continuous. The value at any point of discontinuity is to be taken to be the mean of the values on either side, and $f(0)$ is to be replaced* by $\frac{1}{2}f(0)$. If $f(\lambda)$ vanishes unless λ lies between certain limits, we may reduce the integral to one taken between these limits, and in the Series B take only those terms the arguments of which lie between these limits.†

6. Putting $f(\lambda) \equiv 2/\pi\sqrt{(1-\lambda^2)}$, $u = pa$ and $\alpha = \sin\theta$,

$$\sum_1^\infty \cos(npa\sin\theta)\mathrm{J}_0\, npa$$
$$= \frac{1}{pa}\Sigma\left[\frac{1}{\sqrt{\{1-(2m\pi/pa+\sin\theta)^2\}}}+\frac{1}{\sqrt{\{1-(2m\pi/pa-\sin\theta)^2\}}}\right]-\tfrac{1}{2}.$$

In the case of the Υ_0 functions, take the integral in the first instance from 0 to κ. Then,

$$\frac{2}{\pi}\sum_1^\infty \cos(npa\sin\theta)\int_1^\kappa \frac{\cos\lambda npa}{\sqrt{(\lambda^2-1}}\,d\lambda$$
$$= \frac{1}{pa}\Sigma\left[\frac{1}{\sqrt{\{(2m\pi/pa+\sin\theta)^2-1\}}}+\frac{1}{\sqrt{\{(2m\pi/pa-\sin\theta)^2-1\}}}\right]-\frac{\log 2\kappa}{\pi},$$

but

$$\frac{\log 2\kappa}{\pi} = \frac{1}{\pi}\log 2+\left(1+\tfrac{1}{2}+\ldots+\frac{1}{\kappa}-\gamma\right)/\pi$$
$$= \frac{-\gamma+\log 2}{\pi}+\frac{1}{\pi}+\frac{1}{2\pi}+\ldots+\frac{1}{\kappa\pi},$$

where we may add or take away any finite number of terms at the end of the series. Hence, making $\kappa = \infty$, our sum is

$$\Sigma\left[\frac{1}{\sqrt{\{(2m\pi+pa\sin\theta)^2-p^2a^2\}}}+\frac{1}{\sqrt{\{(2m\pi-pa\sin\theta)^2-p^2a^2\}}}-\frac{1}{m\pi}\right]$$
$$+\frac{\gamma-\log 2}{\pi}.$$

Equating now the force due to the incident waves to the sum of the other forces and the reversed acceleration multiplied by the mass, we have

$$\begin{aligned}\sigma = \mathrm{A}\Big\{ & -b-kp^2\sigma^2 i-2\sigma^2p^2\pi i\Sigma\left[\{p^2a^2-(2m\pi+pa\sin\theta)^2\}^{-\frac{1}{2}}\right.\\ & \left.+\{p^2a^2-(2m\pi-pa\sin\theta)^2\}^{-\frac{1}{2}}\right]+\sigma^2p^2\pi i\\ & +2\sigma^2p^2\pi\Sigma\left[\{(2m\pi+pa\sin\theta)^2-p^2a^2\}^{-\frac{1}{2}}+\{(2m\pi-pa\sin\theta)^2-p^2a^2\}^{-\frac{1}{2}}-\frac{1}{m\pi}\right]\\ & +2\sigma^2p^2(\gamma-\log 2)\\ & +p^2\mathrm{M}\Big\},\end{aligned} \qquad (1)$$

where M is the mass per unit length of a rod.

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* *Cf.* Dirichlet's investigation of Fourier's Series.

† If we take $f(\lambda) \equiv \exp(-k\lambda^2)$, we get a theorem in Elliptic Functions.

The summations are in each case for integral values of m. In the first case such values are to be taken as make the ( ) neither negative nor greater than pa. In the second case the ( ) are not to be less than pa, the m in $1/m\pi$ is to range from 1 to ∞ and each [ ] is to be considered as one term (this will require the omission of one or two of the three terms from some of the [ ] at the beginning of the series).

7. Now let p increase through a value

$$p = 2m\pi/a\,(1 \pm \sin\theta). \tag{2}$$

The first Σ becomes infinite when p is infinitesimally greater than this value, and the second Σ becomes infinite when p is infinitesimally less than it. In either case A vanishes. Hence each of the spectra composing the diffracted light will have a dark line for any such value of p.

Again, if b is large, so that the natural period of a rod is small, there will be a value of p, rather less than that which makes the second Σ infinite, for which the real part of the coefficient of A vanishes. In this case the value of A will be exceptionally large. We have, in fact, a case of resonance. Hence in each spectrum there will be a bright line near to the dark one and on the red side of it.

8. We must now consider the case where the electric force in the incident waves is perpendicular to the rods. First let the rods be capable of vibration only in their own plane. The electric force due to a rod is now

$$\left(p^2+\frac{d^2}{dx^2}\right)2\sigma\mathrm{A}\int_1^\infty \frac{\exp\left(-ipru\right)}{\surd\left(u^2-1\right)}\,du.$$

This is to be multiplied by $\cos\left(pr\sin\theta\right)$ and summed for all values of r given by $r = na$ (excluding $n = 0$). Now if $K(pr)$ is any Bessel's function of zero order, $\left(p^2+\frac{d^2}{dx^2}\right)K\left(pr\right)$ is equal to $r^{-\frac{3}{2}}$ multiplied by a finite quantity (for the case $r = 0$ is excluded). Hence the series in question converges and is finite for all values of p. Hence there are no singularities of the kind that were discussed in § 7.

9. Next let the rods be capable of vibration only at right angles to their plane, the incident waves being as before. The force due to a rod is now

$$\left(p^2+\frac{d^2}{dy^2}\right)\Pi = -\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)\Pi = -\frac{d^2\Pi}{dx^2},$$

where Π is now $2\sigma\mathrm{A}\int_1^\infty \frac{e^{-ipru}}{\surd(u^2-1)}\,du.$ Therefore we get the force in question by subtracting that found in § 8 from that found in § 6.

The former force gives rise to no singularities, and those that the latter gives rise to have already been discussed. There is, however, one new feature, for the left-hand side of equation (1) now is $\sigma \sin \theta$, with the result that there are no diffracted waves if $\theta = 0$.

Finally, let the rods be free to vibrate in any direction. If the force of restitution is opposite to the displacement the diffracted wave is the sum of those found in §§ 8, 9, and no new singularities occur. If the force of restitution is a linear vector function of the displacement the same result seems to hold, but there may be a diffracted wave in the case of perpendicular incidence.

10. We must next consider the effect of change in the angle of incidence of the light. As θ increases one of the values of p given by (2) increases and the other decreases; hence one set of dark lines moves towards the blue end of the spectrum and the other towards the red, in each case accompanied more or less closely by the corresponding bright lines.

11. We cannot obtain mechanical systems that have frequencies comparable with that of light. However, when a conducting rod is placed in an electric field perpendicular to it, a displacement of electricity occurs in it which makes it behave in much the same way as if it were a rigidly electrified rod displaced bodily, and the frequency will be comparable with that of light if the diameter of the rod is comparable with the wave-length of light. The damping due to radiation is also high, and therefore in the case of a wire grating exposed to waves in which the electric force is perpendicular to the wires, we should expect to find the dark lines but not the bright ones. The kind of grating that seems most favourable to the production of the latter is one in which the grooves are narrow cracks, such as a diamond makes in glass when used to "cut" it. Possibly, however, the cracks in a grating made in the way suggested would be so narrow that there would be no appreciable reflection at them, even at very oblique incidences.